

Topological Entropy and Universal Cellular Automata

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Abstract. Modern cellular automata theory distinguishes two genuinely different notions of computational capacity: local universality and global universality. The latter is known to be a strictly stronger property, as no reversible cellular automaton can be globally universal, whereas reversible locally universal cellular automata do exist. Our results further explore the gap between these fundamental computational classes, for which general structural results remain scarce. We introduce a new independent separation through topological entropy. We prove that, in dimension one, there exist locally universal cellular automata with zero topological entropy, while no globally universal cellular automaton can have zero topological entropy. In each dimension $d \geq 2$, every globally universal cellular automaton must even have infinite topological entropy. Topological entropy thus links an asymptotic property from topological dynamics with the computational-theoretic notions of universality for cellular automata.

Keywords: Cellular Automata · Computational Capacity · Global, Intrinsic and Local Universality · Topological Entropy

1 Introduction

The computational power of cellular automata (CA) has been the subject of extensive study ever since von Neumann’s seminal construction of a universal self-replicating automaton [1]. In modern terminology, two different notions of computational universality in cellular automata have emerged [2]. On one hand, *local universality*, or *Turing completeness*: a CA is locally universal if it can simulate a universal Turing machine within a finite, spatially localized region. On the other hand, the stronger notion of *global universality* (also studied under the name of *intrinsic universality*): a CA is globally universal if it can simulate the full spacetime diagram of any other CA, up to geometric transformations such as packing cells into macro-cells, spatial shifts, and time delays. In both cases, the simulation is mediated by translation maps between the simulated and simulating systems, importantly these maps must be computationally restricted, so that the computation is genuinely performed by the CA dynamics rather than by the encoding or decoding procedure itself [2].

Global universality is known to be strictly stronger than local universality: reversible cellular automata can be locally universal [3,4], but they cannot be globally universal, since they cannot globally simulate irreversible CAs [5].

Recently, this gap was further explored by Cotler et al. [2], where locally but not globally universal CAs capable of non-trivial self-replication were constructed. It is argued that CAs allowing for universal self-replicators form an intermediate class within this gap, revealing another notion of universality between local and global universality.

We analyze the separation of local and global universality further by proving that topological entropy also separates these fundamental computational classes, bridging topological dynamics and the computational theory of cellular automata.

Theorem 1 (Entropy separates local and global universality).

There exists a one-dimensional locally universal cellular automaton with topological entropy zero.

On the other hand, no one-dimensional globally universal cellular automaton has zero topological entropy. Moreover, in every dimension $d \geq 2$, every globally universal cellular automaton has infinite topological entropy.

2 Global Universality Result

Global universality captures the idea that a CA can reproduce the full space-time diagrams on the infinite grid $\mathbb{Z}^d$ of any other cellular automaton of the same dimension, up to simple geometric transformations of space and time. In essence, a CA $\mathcal{A} = (S^{\mathbb{Z}^d}, F)$ globally simulates another CA $\mathcal{B} = (T^{\mathbb{Z}^d}, G)$ if there exist suitable transformations $\tilde{G}$ of G and $\tilde{F}$ of F , together with a computationally feasible partial decoder $\mathcal{D}$, such that:

$$\mathcal{D} \circ \tilde{F} \circ \mathcal{D}^{-1}(c) = \tilde{G}(c) \quad \text{for all } c \in T^{\mathbb{Z}^d}.$$

More concretely, the simulated system is obtained as a computationally reasonable factor of a suitable subsystem of a packed configuration space of the simulating system, with the simulating dynamics replaced by a shifted and time-delayed map $\sigma_v \circ F^n$. We refer to [2] for the precise definition used here.

Connection to Intrinsic Universality: Closely related notions have been extensively studied under the name of *intrinsic simulation* or *intrinsic universality* [6,7,8,9,10,11,12]. These notions are the same in spirit, but their differences lie in the precise class of allowed transformations, for instance which block encodings, shifts, rescalings, or subsystems are permitted.

Crucially, we want to emphasize that our arguments do not depend on these definitional choices. The entropy estimates used below are stable under the operations common to the standard simulation notions: passing to shift-invariant subsystems, taking shift-commuting factors, iterating in time, packing cells into macro-cells, and composing with spatial shifts. Consequently, although we state the results in the language of global simulation, the very same proofs apply to the standard formalized notions of intrinsic simulation after replacing the allowed transformations accordingly.

Proof Idea: We observe how, for CAs, zero topological entropy is preserved under factors, subsystems, iterations, and packings (see for example [13]), whereas its behaviour under composition with a spatial shift is not governed by a simple rule. This directly motivates the study of minimal directed entropy instead. Similarly to directional entropy first introduced by Milnor [14], we define the *minimal directional entropy* of a CA $\mathcal{B} = (T^{\mathbb{Z}^d}, G)$ to be:

$$h_{\text{dir}}(\mathcal{B}) := \inf_{(v,n) \in \mathbb{Z}^d \times \mathbb{N}_{>0}} h_{\text{top}}(\sigma_v \circ G^n).$$

The key observation is that, in every dimension, certain linear sensitive cellular automata over the field $\mathbb{Z}_2$ are asymptotically complex in all directions. Let $d \in \mathbb{N}$; for example, the d -dimensional extension of Rule 90 is defined as:

$$\mathcal{B}_d := (\mathbb{F}_2^{\mathbb{Z}^d}, F_d), \quad F_d(c)(v) := c(v - e_1) + c(v + e_1) \pmod{2}.$$

As a consequence of Theorems 2 and 3 of D'amico et al. [15] and Theorem 3.1 of Manzini et al. [16], we have that $(\mathbb{F}_2^{\mathbb{Z}^d}, \sigma_v \circ F_d^n)$ is a linear and sensitive cellular automaton for every $d \in \mathbb{N}$ and $(v, n) \in \mathbb{Z}^d \times \mathbb{N}_{>0}$, which yields the following result:

Lemma 1. *In dimension one, $h_{\text{dir}}(\mathcal{B}_1) > 0$, and $h_{\text{dir}}(\mathcal{B}_d) = \infty$ for every dimension $d \geq 2$.*

Lemma 1 thus allows us to prove the following central proposition:

Proposition 1. *If a d -dimensional CA $\mathcal{A} = (S^{\mathbb{Z}^d}, F)$ globally simulates $\mathcal{B}_d$, then:*

$$h_{\text{top}}(\mathcal{A}) = \infty \quad \text{whenever } d \geq 2, \quad 0 < h_{\text{dir}}(\mathcal{A}) \leq h_{\text{top}}(\mathcal{A}) \quad \text{when } d = 1.$$

3 Local Universality Result

In spirit, a CA is locally universal, or Turing complete, if it can simulate a universal Turing machine in a finite localized region. As noted by Ollinger [12], however, formally defining local universality proves to be challenging. Only recently has there been a new proposal for a formal definition thereof by Cotler et al. [2]. Building on their definition, we construct the following:

Theorem 2. *For any Turing machine $\mathcal{T} = (\Sigma, Q, q_{\text{halt}}, \delta)$, there exists a one-dimensional cellular automaton $\mathcal{A}_{\mathcal{T}} = (S^{\mathbb{Z}}, F_{\mathcal{T}})$ locally simulating $\mathcal{T}$ so that $h_{\text{dir}}(\mathcal{A}_{\mathcal{T}}) = h_{\text{top}}(\mathcal{A}_{\mathcal{T}}) = 0$.*

In particular, there exists a one-dimensional locally universal cellular automaton with entropy zero.

Proof Idea: We build on Kůrka's interpretation of Turing machines (TM) with moving head as zero-entropy dynamical systems [17,18]. We adapt such a system into a CA $\mathcal{A}_{\mathcal{T}}$ whose local rule simulates the TM $\mathcal{T}$ on configurations with a single head, while crucially enforcing configurations with more than one head to eventually converge to a uniform quiescent state, which allows us to prove $h_{\text{top}}(\mathcal{A}_{\mathcal{T}}) = 0$.

Remark 1. We conjecture, though it is not yet proven, that a construction similar to the above yields the same result for local universality in arbitrary dimensions $d \geq 2$, and not just in dimension one.

Remark 2. In dimension one, our argument proves the even stronger statement that a globally universal CA must have positive *minimal directional* entropy. Hence, as an example, the CA implementing the one dimensional shift cannot be globally universal, although it has positive entropy.

Whether, in dimensions $d \geq 2$, a similar strengthening holds, namely whether global universality implies infinite *minimal directional* entropy, remains currently open.

Remark 3. We conclude by adding how topological entropy truly is a new separation between local and global universality, as reversibility and zero entropy are independent in dimension one, and reversibility and infinite entropy are independent in all dimensions $d \geq 2$.

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